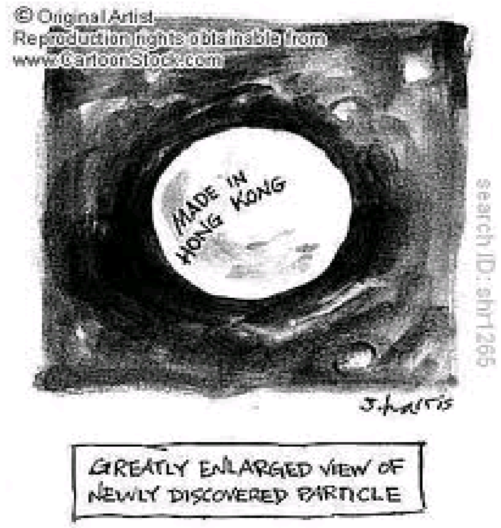

Stochastic Control Applications in Life Insurance

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IME 2012, Hong Kong

Outline:

- Quadratic optimization - pension fund control classics and beyond
- Utility optimization in with-profit life insurance - the company's decision making
- Utility optimization in life insurance - the individual's decisions making
- Optimal investment of a pension fund - some modern thoughts and methods
- Other objectives

- Quadratic optimization - pension fund control classic and beyond

$$
\begin{aligned}
dX(t) &= \underbrace{p(t)\,dt}_{\text{premiums}} - \underbrace{b(t)\,dt + \beta(t)\,d\overline{W}(t)}_{\text{benefits}} \\
&\quad + \underbrace{X(t)\left(\left(r + \pi(t)(\alpha - r)\right)dt + \pi(t)\,\sigma dW(t)\right)}_{\text{capital gains}}
\end{aligned}
$$

Defined benefits: For fixed b, β

$$
\min_{p,\pi} E\left[\int_0^\infty e^{-\rho t}\left\{a\left(p(t) - \underbrace{\widehat{p}(t)}_{\text{premium target}}\right)^2 + \left(X(t) - \underbrace{\widehat{x}(t)}_{\text{local wealth target}}\right)^2\right\}\right]
$$

$$
V(t,x) = \min_{p,\pi} E_{t,x}\left[\int_t^\infty e^{-\rho(s-t)}\left\{a\left(p(s) - \widehat{p}(s)\right)^2 + \left(X(s) - \widehat{x}(s)\right)^2\right\}\right]
$$

Dynamic programming converts a time-global optimization problem into a time-local one:

$$V_t = \max_{p,\pi} \left\{ \begin{array}{c} \rho V - (p - b(t) + (r + \pi(\alpha - r))x) V_x - \frac{1}{2}\left(\beta^2(t) + \pi^2\sigma^2 x^2\right) V_{xx} \\ - \left(a(p - \widehat{p}(t))^2 + (x - \widehat{x}(t))^2\right) \end{array} \right\}$$

Solutions

$$\begin{aligned} p^*(t,x) &= \underbrace{\widehat{p}(t) - f(t)\left(x - \underbrace{g(t)}_{\text{global wealth target}}\right)}_{\text{linear regulation}} \\ \pi^*(t,x) &= -k(x - g(t)) \end{aligned}$$

Related to dynamic mean-variance optimization

$$dX(t) = X(t)\left((r + \pi(t)(\alpha - r))\,dt + \pi(t)\,\sigma dW(t)\right)$$

$$\begin{aligned}\min_{\pi: E[X(T)] \geq m} Var[X(T)] &= \min_{\pi: E[X(T)] \geq m} E\left[(X(T) - E[X(T)])^2\right] \\ &= \min_{\pi: E[X^\lambda(T)] = m} E\left[\left(X^\lambda(T) - m\right)^2 + \lambda\left(X^\lambda(T) - m\right)\right]\end{aligned}$$

$$V(t,x) = \min_{\pi: E[X^\lambda(T)] = m} E_{t,x}\left[\left(X^\lambda(T) - m\right)^2 + \lambda\left(X^\lambda(T) - m\right)\right]$$

For $m > x_0 e^{rT}$:

$$\pi^*(t,x) = k\left(\underbrace{g(t)}_{\text{global wealth target}} - x\right)$$

By modern methods in portfolio optimization one can discuss problems like

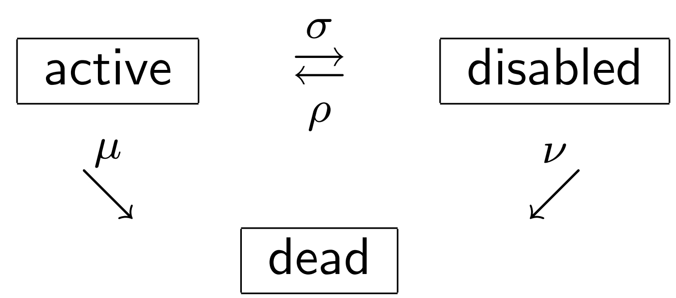
$$\min \left\{ Var_{t,x}\left[X\left(T\right)\right] - E_{t,x}\left[X\left(T\right)\right] \right\}$$
$$\min E_{t,x}\left[\left(X\left(T\right) - xe^{\rho(T-t)} \right)^2 \right]$$
$$\min \left\{ \frac{1}{x} Var_{t,x}\left[X\left(T\right)\right] - E_{t,x}\left[X\left(T\right)\right] \right\}$$
$$\min \left\{ \sqrt{Var_{t,x}\left[X\left(T\right)\right]} - E_{t,x}\left[X\left(T\right)\right] \right\}$$

- Important technical questions on time-consistency arise because of non-linear functions like $\left(E_{t,x}\left[X\left(T\right)\right]\right)^2$ in $Var_{t,x}\left[X\left(T\right)\right]$ and the appearance of (t, x) 'outside' the conditioning

- Important future research in integration of these modern methods and pension fund control classics

- Utility optimization in with-profit life insurance - the company's decision making

- Utility optimization in life insurance - the individual's decisions making



$$
\begin{aligned}
dX(t) &= d\Big(\underbrace{A(t) - L(t)}_{\text{assets minus liabilities}}\Big) \\
&= \underbrace{dC(t)}_{\text{systematic surplus contributions}} - \underbrace{dD(t)}_{\text{dividends/bonus}} + \text{'noise from } C\text{'} \\
&\quad + \underbrace{X(t)\left(\left(r + \pi(t)(\alpha - r)\right)dt + \pi(t)\,\sigma dW(t)\right)}_{\text{capital gains}}
\end{aligned}
$$

$$
\begin{aligned}
dC(t) &= \sum_j I^j(t) \underbrace{c^j(t)\,dt}_{\text{surplus contributions: set to 0 by realistic valuation of liabilities}} \\
dD(t) &= \sum_j I^j(t) \underbrace{\delta^j(t)\,dt}_{\text{dividend payments e.g. } X \text{ dependent}}
\end{aligned}
$$

The dividend/bonus reserve, the free reserves

$$R(t,x) = E^Q_{t,x}\left[\int_t^\infty e^{-\int_t^s r} dD(s)\right] = E^Q_{t,x}\left[\int_t^\infty e^{-\int_t^s r} \sum_j I^j(s)\,\delta^j(s)\,ds\right]$$

The dividend plan is financially 'fair' if $R(t,x) = x$. But how should one choose among different 'fair' dividend plans? The utility reserve

$$\begin{aligned} U(t,x) &= E_{t,x}\left[\int_t^\infty e^{-\int_t^s \rho} \sum_j I^j(s) \underbrace{w^j(s)\,u\left(\delta^j(s)\right)}_{\text{utility of dividends}} ds\right] \\ V(t,x) &= \max_{\delta,\pi} U(t,x) \end{aligned}$$

Utility optimization in life insurance - the individual's decisions making

$$
\begin{aligned}
dX(t) &= \underbrace{dA(t)}_{\text{labor income}} - \underbrace{dC(t)}_{\text{consumption}} \\
&\quad + \underbrace{X(t)\left(\left(r + \pi(t)(\alpha - r)\right)dt + \pi(t)\sigma dW(t)\right)}_{\text{capital gains}}
\end{aligned}
$$

$$
\begin{aligned}
dA(t) &= \sum_j I^j(t) \underbrace{a^j(t)\,dt}_{\text{statewise income rates}} \\
dC(t) &= \sum_j I^j(t) \underbrace{c^j(t)\,dt}_{\text{statewise consumption rates}}
\end{aligned}
$$

In the presence of labor income, we need access to an insurance market (to achieve elegant solutions)

We introduce controllable 'sums at risk'

$$dX(t) = ... + \sum_{k \neq j} \underbrace{l^{jk}(t)}_{\text{controllable loss upon transition}} \left(dN^{jk}(t) - \underbrace{\widehat{\mu}^{jk}(t)\,dt}_{\text{pricing basis}} \right)$$

There is an interesting exercise in translating the optimal 'sums at risk' to the optimal classical coverages, but this not our focus here

$$V(t,x) = \max_{l,c,\pi} E_{t,x} \left[\int_t^\infty e^{-\int_t^s \rho} \sum_j I^j(s) \underbrace{w^j(s)\, u\left(c^j(s)\right)}_{\text{utility of consumption}} ds \right]$$

The solution to the company's decision problem (no surplus contribution - or no income)

The power utility solution generalizes Merton's consumption-investment results

$$\delta^*(t,x) = \frac{x}{f^j(t)}$$
$$\pi^*(t,x) = \frac{1}{\gamma}\frac{\alpha - r}{\sigma^2} \text{ (CPPI)}$$

Remarks:

What if w are the guaranteed payments? Then f is (some) technical value of guaranteed benefits (interpretation?)

What if dividends are used to increase guaranteed benefits? Then δ follows a barrier strategy (interpretation?)

What if utility is not time-additive???

The solution to the individual's decision problem (with income and insurance - or with surplus contributions and hedging of individual risk)

The power utility solution generalizes Merton's consumption-investment results for an investor with income

$$
\begin{aligned}
c^{*j}(t,x) &= \frac{x + \overbrace{g^j(t)}^{\text{human wealth}}}{f^j(t)} \\
g^j(t) &= \widehat{E}_{t,j}\left[\int_t^n e^{-\int_t^s r} \underbrace{dA(s)}_{\text{income around } s}\right] \\
x + g^k(t) + l^{*jk}(t) &= \left(x + g^j(t)\right) * \text{function of } \left(\lambda, \frac{\widehat{\mu}}{\mu}\right)
\end{aligned}
$$

Can be used to provide 'good advice' for changing positions or optimal (buy-and-hold) product design

The simplest example: Depending on risk aversion γ, time preferences δ, and pricing kernel $\frac{\widehat{\mu}}{\mu}$: An optimal consumption rule produces directly an optimal annuity benefit profile and an optimal premium profile by

$$
\begin{aligned}
\frac{d}{dt}c(t) &= c(t)\, f\left(\gamma, \delta, \frac{\widehat{\mu}}{\mu}\right) \\
\underbrace{c(t)}_{\text{consumption}} &= \underbrace{b(t)}_{\text{net insurance benefits}} + \underbrace{a(s)}_{\text{income}}
\end{aligned}
$$

What if utility is not additive???

Many interesting variations of these problems (constraints on processes, controls, completeness)

- Optimal investment of a pension fund - some modern thoughts and methods

Portfolio optimization

$$dX(t) = X(t)\left(\left(r + \pi(t)(\alpha - r)\right)dt + \pi(t)\sigma dW(t)\right)$$

What should we optimize if two members with different preferences share the pot equally at the end?

Make two individual accounts and let them invest separately! No, this does not answer the question! (why?)

Then what? Maximize added utility?

$$\max_{\pi} E\left[u\left(\frac{X(T)}{2}\right) + v\left(\frac{X(T)}{2}\right)\right]?$$

Support from economic equilibrium theory? But also problematic?

What else? Maximize added certainty equivalents?

$$\max_{\pi}\left\{\underbrace{u^{-1}\left(E\left[u\left(\frac{X(T)}{2}\right)\right]\right)}_{\text{first certainty equivalent}} + \underbrace{v^{-1}\left(E\left[v\left(\frac{X(T)}{2}\right)\right]\right)}_{\text{second certainty equivalent}}\right\}$$

Now we are back with non-linearity problems 'similar to' the mean-variance case. One possibility:

$$V(t,x) = \max_{\pi} \left\{ \underbrace{u^{-1}\left(E_{t,x}\left[u\left(\frac{X(T)}{2}\right)\right]\right)}_{\text{first certainty equivalent}} + \underbrace{v^{-1}\left(E_{t,x}\left[v\left(\frac{X(T)}{2}\right)\right]\right)}_{\text{second certainty equivalent}} \right\}$$

Nice solution for e.g. power utility

$$\pi = \frac{1}{\gamma(t)} \frac{\alpha - r}{\sigma^2}$$

where γ is a weighted average of individual γs with time-to-maturity dependent weights

Can this inspire to an attack on 'non-additive utility'? If we add certainty equivalence here, why not elsewhere?

$$
\begin{aligned}
V(t,x) &= \max_{c,\pi} E_{t,x}\left[\int_t^{\infty} e^{-\int_t^s \rho} c^{\gamma}(s)\, ds\right] \\
(\text{for } \gamma=\phi) &= \max_{c,\pi}\left\{\int_t^{\infty} e^{-\int_t^s \rho}\left(\underbrace{\left(E_{t,x}\left[c^{\gamma}(s)\right]\right)^{\frac{1}{\gamma}}}_{\text{certainty equivalent}}\right)^{\phi} ds\right\}
\end{aligned}
$$

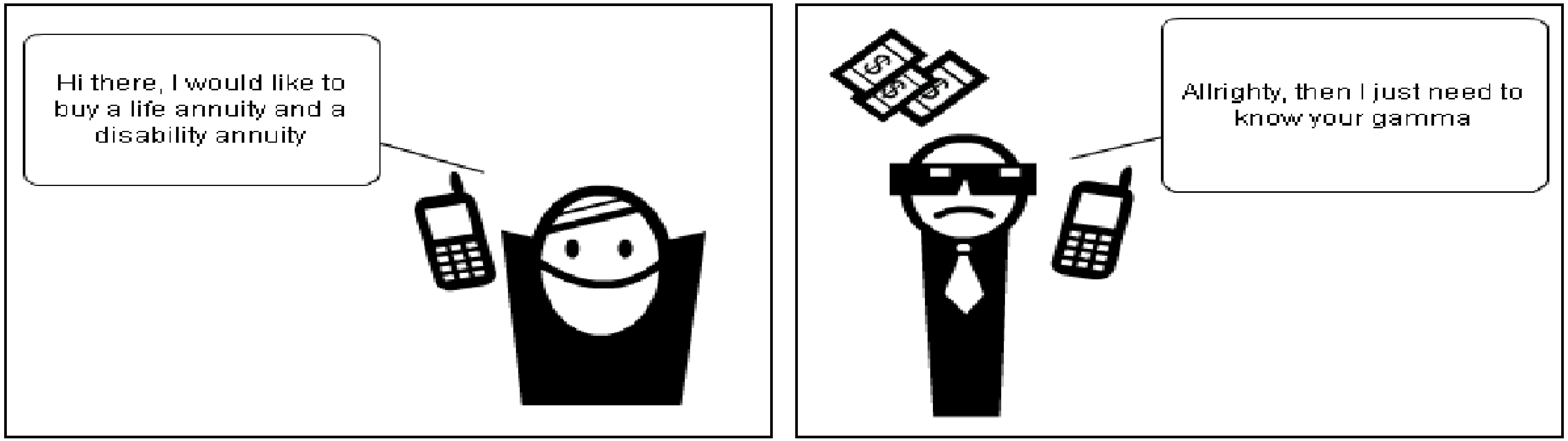
But what happens if $\phi \neq \gamma$? And how should we interpret ϕ?

ϕ is related to notion of 'elasticity of intertemporal substitution'

With modern methods: *Closely* related to the notion of recursive utility!

A lot to be done to understand these things better and integrate insurance decisions

- Other objectives
  - personal ruin probability minimization (is this a more *objective* objective than utility maximization? Why/why not?)
    * Set a consumption level that you cannot finance until death almost surely in a given market
    * Invest and/or insure (life annuities) to minimize the probability of personal ruin
  - Controlling variability of consumption rather than level of consumption
    * For $dc\left(t\right) = a\left(t\right)dt$ control a, e.g. linearly, rather than c?


Hi there, I would like to buy a life annuity and a disability annuity
Allrighty, then I just need to know your gamma